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ON THE EQUATIONS OF ELASTIC MATERIALS WITH MICRO-STRUCTURE

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Abstract—In the classical theory of elasticity it is possible to reduce the displacement equations of motion to Lamé's form by means of the Helmholtz resolution. In this paper the analogous reduction is effected for equations, derived recently [1], of an isotropic, elastic material with micro-structure.

INTRODUCTION

IN THE classical theory of elasticity, the displacement-equation of motion,

$$(\lambda+2\mu)\nabla\nabla\cdot\mathbf{u}-\mu\nabla\times\nabla\times\mathbf{u}+\mathbf{f}=\rho\ddot{\mathbf{u}}, \tag{1}$$

is reduced, by means of the Helmholtz resolutions

$$\mathbf{u}=\nabla\varphi_1+\nabla\times\mathbf{H}_1, \qquad \nabla\cdot\mathbf{H}_1=0, \tag{2a}$$

$$\mathbf{f}=\nabla\varphi_1^*+\nabla\times\mathbf{H}_1^*, \qquad \nabla\cdot\mathbf{H}_1^*=0, \tag{2b}$$

to Lamé's form:

$$(\lambda+2\mu)\nabla^2\varphi_1+\varphi_1^*=\rho\ddot{\varphi}_1, \tag{3a}$$

$$\mu\nabla^2\mathbf{H}_1+\mathbf{H}_1^*=\rho\ddot{\mathbf{H}}_1. \tag{3b}$$

In this paper, the analogous reduction is effected for equations, derived recently [1], of an isotropic, elastic material with micro-structure. The new equations have, in addition to the displacement vector $\mathbf{u}$, a second dependent variable: the micro-deformation dyadic $\boldsymbol{\psi}$. The variables $\mathbf{u}$ and $\boldsymbol{\psi}$ are governed by the equations†

$$\begin{aligned}&(\mu+2g_2+b_2)\nabla^2\mathbf{u}+(\lambda+\mu+2g_1+2g_2+b_1+b_3)\nabla\nabla\cdot\mathbf{u}\\&\qquad-(g_1+b_1)\nabla(\mathbf{I}:\boldsymbol{\psi})-(g_2+b_2)\nabla\cdot\boldsymbol{\psi}-(g_2+b_3)\boldsymbol{\psi}\cdot\nabla+\mathbf{f}=\rho\ddot{\mathbf{u}},\end{aligned} \tag{4a}$$

$$\begin{aligned}&(a_1+a_5)[\mathbf{I}\nabla\cdot\boldsymbol{\psi}\cdot\nabla+\nabla\nabla(\mathbf{I}:\boldsymbol{\psi})]+(a_2+a_{11})(\nabla\cdot\boldsymbol{\psi}\nabla+\nabla\boldsymbol{\psi}\cdot\nabla)\\&\quad+(a_3+a_{14})\nabla\nabla\cdot\boldsymbol{\psi}+a_4\mathbf{I}\nabla^2(\mathbf{I}:\boldsymbol{\psi})+(a_8+a_{15})\boldsymbol{\psi}\cdot\nabla\nabla+a_{10}\nabla^2\boldsymbol{\psi}\\&\qquad+a_{13}\nabla^2\boldsymbol{\psi}_C+g_1\mathbf{I}\nabla\cdot\mathbf{u}+g_2(\nabla\mathbf{u}+\mathbf{u}\nabla)+b_1\mathbf{I}(\nabla\cdot\mathbf{u}+\mathbf{I}:\boldsymbol{\psi})\\&\qquad\qquad+b_2(\nabla\mathbf{u}-\boldsymbol{\psi})+b_3(\mathbf{u}\nabla-\boldsymbol{\psi}_C)+\boldsymbol{\Phi}=\tfrac{1}{3}\rho' d^2\ddot{\boldsymbol{\psi}},\end{aligned} \tag{4b}$$

where $\boldsymbol{\psi}_C$ is the conjugate of $\boldsymbol{\psi}$, $\mathbf{I}$ is the idemfactor, $\boldsymbol{\Phi}$ is the body double force dyadic, ρ' is the mass density of the micro-material, d is the half-length of the unit micro-cell and a_i, b_i, g_i are material constants.

Whereas (1) is equivalent to three scalar equations, (4) are equivalent to twelve scalar equations.

† See [1], equations (6.1) and (6.2).

ALTERNATIVE FORM OF EQUATIONS

A complete dyadic, $\mathbf{\Delta}$, may be resolved into the sum of its symmetric part $\mathbf{\Delta}^S$ and antisymmetric part $\mathbf{\Delta}^A$:

$$\mathbf{\Delta} = \mathbf{\Delta}^S+\mathbf{\Delta}^A, \qquad \mathbf{\Delta}^S = \tfrac{1}{2}(\mathbf{\Delta}+\mathbf{\Delta}_C), \qquad \mathbf{\Delta}^A = \tfrac{1}{2}(\mathbf{\Delta}-\mathbf{\Delta}_C).$$

The symmetric part may be resolved, further, into the sum of its deviator $\mathbf{\Delta}^{SD}$ and its spherical part $\mathbf{I}\Delta_S$:

$$\mathbf{\Delta}^S = \mathbf{\Delta}^{SD}+\mathbf{I}\Delta_S, \qquad \mathbf{\Delta}^{SD} = \mathbf{\Delta}^S-\tfrac{1}{3}\mathbf{II}:\mathbf{\Delta}, \qquad \Delta_S = \tfrac{1}{3}\mathbf{I}:\mathbf{\Delta}.$$

Thus,

$$\mathbf{\Delta} = \mathbf{\Delta}^{SD}+\mathbf{I}\Delta_S+\mathbf{\Delta}^A. \tag{5}$$

Necessary and sufficient conditions for $\mathbf{\Delta} = 0$ are

$$\mathbf{\Delta}^{SD} = 0, \qquad \Delta_S = 0, \qquad \mathbf{\Delta}^A = 0. \tag{6}$$

Upon resolving the dyadics $\boldsymbol{\psi}$ and $\mathbf{\Phi}$, in (4), in accordance with (5), and applying (6) to the resulting equations, the following equations on $\mathbf{u}$, $\boldsymbol{\psi}^{SD}$, ψ_S and $\boldsymbol{\psi}^A$ are obtained:

$$k_{11}\nabla\nabla\cdot\mathbf{u}-k_{12}\nabla\cdot\boldsymbol{\psi}^{SD}-k_{13}\nabla\psi_S-\bar{k}_{11}\nabla\times\nabla\times\mathbf{u}-\bar{k}_{13}\nabla\cdot\boldsymbol{\psi}^A_C+\mathbf{f} = \rho\ddot{\mathbf{u}}, \tag{7a}$$

$$k_{21}(\nabla\mathbf{u})^{SD}+k_{22}(\nabla\nabla\cdot\boldsymbol{\psi}^{SD})^{SD}-\tfrac{2}{3}k'_{22}\boldsymbol{\psi}^{SD}+k_{23}(\nabla\nabla\psi_S)^D$$

$$+\bar{k}_{23}(\nabla\nabla\cdot\boldsymbol{\psi}^A)^S+(a_{10}+a_{13})[\nabla^2\boldsymbol{\psi}^{SD}-\tfrac{3}{2}(\nabla\nabla\cdot\boldsymbol{\psi}^{SD})^{SD}]+\mathbf{\Phi}^{SD} = \tfrac{1}{3}\rho' d^2\ddot{\boldsymbol{\psi}}^{SD}, \tag{7b}$$

$$k_{31}\nabla\cdot\mathbf{u}+k_{32}\nabla\cdot\boldsymbol{\psi}^D\cdot\nabla+k_{33}\nabla^2\psi_S-k'_{33}\psi_S+3\Phi_S = \rho' d^2\ddot{\psi}_S, \tag{7c}$$

$$\bar{k}_{31}(\nabla\mathbf{u})^A+\bar{k}_{32}(\nabla\nabla\cdot\boldsymbol{\psi}^{SD})^A+\bar{k}_{33}(\nabla\nabla\cdot\boldsymbol{\psi}^A)^A-\tfrac{1}{2}\bar{k}'_{33}\boldsymbol{\psi}^A$$

$$+(a_{10}-a_{13})[\nabla^2\boldsymbol{\psi}^A-2(\nabla\nabla\cdot\boldsymbol{\psi}^A)^A]+\mathbf{\Phi}^A = \tfrac{1}{3}\rho' d^2\ddot{\boldsymbol{\psi}}^A, \tag{7d}$$

where

$$k_{11} = \lambda+2\mu+2g_1+4g_2+b_1+b_2+b_3,$$
$$k_{22} = 2a_2+a_3+a_8+\tfrac{3}{2}a_{10}+2a_{11}+\tfrac{3}{2}a_{13}+a_{14}+a_{15},$$
$$k_{33} = 6a_1+2a_2+a_3+9a_4+6a_5+a_8+3a_{10}+2a_{11}+3a_{13}+a_{14}+a_{15},$$
$$k_{23} = k_{32} = 3a_1+2a_2+a_3+3a_5+a_8+2a_{11}+a_{14}+a_{15},$$
$$k_{31} = k_{13} = 3g_1+2g_2+3b_1+b_2+b_3,$$
$$k_{12} = k_{21} = 2g_2+b_2+b_3,$$
$$k'_{22} = \tfrac{3}{2}(b_2+b_3),$$
$$k'_{33} = 3(3b_1+b_2+b_3),$$
$$\bar{k}_{11} = \mu+2g_2+b_2,$$
$$\bar{k}_{22} = 2a_2+a_3+a_8+2a_{10}+2a_{11}+2a_{13}+a_{14}+a_{15},$$
$$\bar{k}_{33} = -2a_2+a_3+a_8+2a_{10}-2a_{11}-2a_{13}+a_{14}+a_{15},$$
$$\bar{k}_{23} = \bar{k}_{32} = a_3-a_8+a_{14}-a_{15},$$

$$\bar{k}_{31} = \bar{k}_{13} = b_2 - b_3,$$
$$\bar{k}_{12} = \bar{k}_{21} = 2g_2 + b_2 + b_3,$$
$$\bar{k}'_{22} = 2(b_2 + b_3),$$
$$\bar{k}'_{33} = 2(b_2 - b_3).$$

ANALOGUES OF HELMHOLTZ RESOLUTION

To find a resolution of the dyadic $\boldsymbol{\psi}$ (or $\boldsymbol{\Phi}$) analogous to that of the vector $\mathbf{u}$ (or $\mathbf{f}$) in (2), first define

$$4\pi\boldsymbol{\psi}' \equiv -\int_V r^{-1}\boldsymbol{\psi}_Q \,\mathrm{d}V_Q, \tag{8}$$

where $r\ (= \sqrt{[(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2]})$ is the distance between a field point $P(x, y, z)$ and a source point $Q(\xi, \eta, \zeta)$, $\boldsymbol{\psi}_Q = \boldsymbol{\psi}(\xi, \eta, \zeta)$ and $\mathrm{d}V_Q = \mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta$. Then

$$\boldsymbol{\psi} = \nabla^2\boldsymbol{\psi}' = \nabla\nabla\cdot\boldsymbol{\psi}' - \nabla\times\nabla\times\boldsymbol{\psi}'.$$

Define

$$\mathbf{G}' \equiv \nabla\cdot\boldsymbol{\psi}', \qquad \boldsymbol{\Lambda}' \equiv -\nabla\times\boldsymbol{\psi}'.$$

Then a first resolution is

$$\boldsymbol{\psi} = \nabla\mathbf{G}' + \nabla\times\boldsymbol{\Lambda}', \qquad \nabla\cdot\boldsymbol{\Lambda}' = 0. \tag{9}$$

Now, define

$$4\pi\boldsymbol{\Lambda}'' \equiv -\int_V r^{-1}\boldsymbol{\Lambda}'_Q \,\mathrm{d}V_Q,$$

so that

$$\boldsymbol{\Lambda}' = \nabla^2\boldsymbol{\Lambda}'' = (\boldsymbol{\Lambda}''\cdot\nabla)\nabla - \boldsymbol{\Lambda}''\times\nabla\times\nabla.$$

Define

$$\mathbf{G}'' \equiv \boldsymbol{\Lambda}''\cdot\nabla, \qquad \boldsymbol{\Lambda}''' \equiv -\boldsymbol{\Lambda}''\times\nabla.$$

Then

$$\boldsymbol{\Lambda}' = \mathbf{G}''\nabla + \boldsymbol{\Lambda}'''\times\nabla, \qquad \boldsymbol{\Lambda}'''\cdot\nabla = 0, \qquad \nabla^2\nabla\cdot\boldsymbol{\Lambda}''' = 0. \tag{10}$$

Substituting (10) in (9), we find a second form:

$$\boldsymbol{\psi} = \nabla\mathbf{G}' + \nabla\times(\mathbf{G}''\nabla) + \nabla\times\boldsymbol{\Lambda}'''\times\nabla, \qquad \boldsymbol{\Lambda}'''\cdot\nabla = 0, \qquad \nabla^2\nabla\cdot\boldsymbol{\Lambda}''' = 0. \tag{11}$$

Consider, now, the symmetric part of $\boldsymbol{\psi}$:

$$\boldsymbol{\psi}^S = \nabla\mathbf{G} + \mathbf{G}\nabla + \nabla\times\boldsymbol{\Lambda}\times\nabla, \tag{12}$$

where

$$\mathbf{G} \equiv \tfrac{1}{2}(\mathbf{G}' + \nabla\times\mathbf{G}''), \qquad \boldsymbol{\Lambda} \equiv \tfrac{1}{2}(\boldsymbol{\Lambda}''' + \boldsymbol{\Lambda}'''_C) = \boldsymbol{\Lambda}_C \tag{13}$$

and we note that, in view of the second and third parts of (11),

$$\nabla\cdot\boldsymbol{\Lambda}\cdot\nabla = 0, \qquad \nabla^2\nabla\cdot\boldsymbol{\Lambda} = 0. \tag{14}$$

Further, define

$$4\pi \mathbf{G}''' \equiv -\int_V r^{-1}\mathbf{G}_Q \, \mathrm{d}V_Q.$$

Then

$$\mathbf{G} = \nabla^2\mathbf{G}''' = \nabla\nabla\cdot\mathbf{G}''' - \nabla\times\nabla\times\mathbf{G}'''.$$

Define

$$\chi \equiv 2\nabla\cdot\mathbf{G}''', \qquad \mathbf{H}_2 \equiv -\nabla\times\mathbf{G}'''.$$

Then

$$\mathbf{G} = \tfrac{1}{2}\nabla\chi + \nabla\times\mathbf{H}_2, \qquad \nabla\cdot\mathbf{H}_2 = 0. \tag{15}$$

Upon substituting (15) in (12), we find

$$\boldsymbol{\psi}^S = \nabla\nabla\chi + 2(\nabla\nabla\times)^S\mathbf{H}_2 + \nabla\times\boldsymbol{\Lambda}\times\nabla, \tag{16}$$

$$\nabla\cdot\mathbf{H}_2 = 0, \qquad \boldsymbol{\Lambda} = \boldsymbol{\Lambda}_C, \qquad \nabla\cdot\boldsymbol{\Lambda}\cdot\nabla = 0, \qquad \nabla^2\nabla\cdot\boldsymbol{\Lambda} = 0,$$

where the symbol $(\nabla\nabla\times)^S$ is defined according to

$$(\nabla\nabla\times)^S\mathbf{H} \equiv \tfrac{1}{2}[\nabla(\nabla\times\mathbf{H}) + (\nabla\times\mathbf{H})\nabla].$$

The spherical part of $\boldsymbol{\psi}^S$ is

$$\psi_S \equiv \tfrac{1}{3}\mathbf{I}:\boldsymbol{\psi}^S = \tfrac{1}{3}\nabla^2(\chi - \mathbf{I}:\boldsymbol{\Lambda})$$

and the deviator of $\boldsymbol{\psi}^S$ is

$$\begin{aligned}\boldsymbol{\psi}^{SD} &\equiv \boldsymbol{\psi}^S - \mathbf{I}\psi_S,\\ &= \nabla\nabla\chi + 2(\nabla\nabla\times)^S\mathbf{H}_2 + \nabla\times\boldsymbol{\Lambda}\times\nabla - \tfrac{1}{3}\mathbf{I}\nabla^2(\chi - \mathbf{I}:\boldsymbol{\Lambda}),\\ &= (\nabla\nabla)^D\chi + 2(\nabla\nabla\times)^S\mathbf{H}_2 + (\nabla\times\boldsymbol{\Lambda}\times\nabla)^D,\end{aligned}$$

where $(\,)^D$ designates the deviatoric part.

It is convenient to define

$$\varphi_2 \equiv \tfrac{1}{3}(2\chi + \mathbf{I}:\boldsymbol{\Lambda}), \qquad \varphi_3 \equiv \tfrac{1}{3}(\chi - \mathbf{I}:\boldsymbol{\Lambda}) \tag{17}$$

and the symmetric deviator

$$\boldsymbol{\Gamma} \equiv (\nabla\times\boldsymbol{\Lambda}\times\nabla)^D - \tfrac{1}{2}(\nabla\nabla)^D\varphi_2 + (\nabla\nabla)^D\varphi_3. \tag{18}$$

Then

$$\boldsymbol{\psi}^{SD} = \boldsymbol{\Gamma} + \tfrac{3}{2}(\nabla\nabla)^D\varphi_2 + 2(\nabla\nabla\times)^S\mathbf{H}_2, \qquad \nabla\cdot\mathbf{H}_2 = 0, \tag{19}$$

$$\psi_S = \nabla^2\varphi_3. \tag{20}$$

As for the antisymmetric part of $\boldsymbol{\psi}$, we have, from (11),

$$\boldsymbol{\psi}^A = \nabla\mathbf{F} - \mathbf{F}\nabla + \tfrac{1}{2}\nabla\times(\boldsymbol{\Lambda}''' - \boldsymbol{\Lambda}_C''')\times\nabla, \tag{21}$$

where

$$\mathbf{F} \equiv \tfrac{1}{2}(\mathbf{G}' - \nabla\times\mathbf{G}'').$$

Define

$$4\pi \mathbf{F}' \equiv -\int_V r^{-1}\mathbf{F}_Q \, \mathrm{d}V_Q.$$

Then

$$\mathbf{F} = \nabla^2 \mathbf{F}' = \nabla\nabla \cdot \mathbf{F}' - \nabla \times \nabla \times \mathbf{F}'.$$

Define

$$\chi' \equiv \nabla \cdot \mathbf{F}', \qquad \mathbf{H}_3 \equiv -\nabla \times \mathbf{F}'.$$

Then

$$\mathbf{F} = \nabla\chi' + \nabla \times \mathbf{H}_3, \qquad \nabla \cdot \mathbf{H}_3 = 0 \tag{22}$$

and, from (22) and (21),

$$\boldsymbol{\psi}^A = \nabla(\nabla \times \mathbf{H}_3) - (\nabla \times \mathbf{H}_3)\nabla + \tfrac{1}{2}\nabla \times (\boldsymbol{\Lambda}''' - \boldsymbol{\Lambda}_C''') \times \nabla, \tag{23}$$

$$\nabla \cdot \mathbf{H}_3 = 0, \qquad \nabla^2 \nabla \cdot \boldsymbol{\Lambda}''' = 0, \qquad \boldsymbol{\Lambda}''' \cdot \nabla = 0. \tag{24}$$

Now, in view of (24), we have, from (23),

$$\mathbf{I} \times \mathbf{I} : \boldsymbol{\psi}^A = 2\nabla^2 \mathbf{H}_3 + \nabla \mathrm{Y},$$

where

$$\mathrm{Y} \equiv -\mathbf{I} \times \mathbf{I} \,\vdots\, \nabla \boldsymbol{\Lambda}'''.$$

Then

$$\boldsymbol{\psi}^A = \mathbf{I} \times (\tfrac{1}{2}\nabla \mathrm{Y} + \nabla^2 \mathbf{H}_3), \qquad \nabla \cdot \mathbf{H}_3 = 0. \tag{25}$$

Assembling the results in (19), (20), (25), we have a special form of resolution of $\boldsymbol{\psi}$ suitable for the present purpose:

$$\begin{aligned} \boldsymbol{\psi} &= \boldsymbol{\psi}^{SD} + \psi_S \mathbf{I} + \boldsymbol{\psi}^A, \\ &= \boldsymbol{\Gamma} + \tfrac{3}{2}(\nabla\nabla)^D \varphi_2 + 2(\nabla\nabla \times)^S \mathbf{H}_2 + \mathbf{I}\nabla^2 \varphi_3 + \mathbf{I} \times (\tfrac{1}{2}\nabla \mathrm{Y} + \nabla^2 \mathbf{H}_3), \\ &\nabla \cdot \mathbf{H}_2 = 0, \qquad \nabla \cdot \mathbf{H}_3 = 0, \qquad \boldsymbol{\Gamma} = \boldsymbol{\Gamma}_C, \qquad \mathbf{I} : \boldsymbol{\Gamma} = 0. \end{aligned} \tag{26a}$$

Similarly, we may express the body double force dyadic $\boldsymbol{\Phi}$ as

$$\boldsymbol{\Phi} = \boldsymbol{\Phi}^{SD} + \mathbf{I}\Phi_S + \boldsymbol{\Phi}^A, \tag{26b}$$

where

$$\begin{aligned} \boldsymbol{\Phi}^{SD} &= \boldsymbol{\Gamma}^* + \tfrac{3}{2}(\nabla\nabla)^D \varphi_2^* + 2(\nabla\nabla \times)^S \mathbf{H}_2^*, \qquad \nabla \cdot \mathbf{H}_2^* = 0, \\ \Phi_S &= \nabla^2 \varphi_3^*, \\ \boldsymbol{\Phi}^A &= \mathbf{I} \times (\tfrac{1}{2}\nabla \mathrm{Y}^* + \nabla^2 \mathbf{H}_3^*), \qquad \nabla \cdot \mathbf{H}_3^* = 0, \end{aligned}$$

and φ_2^*, φ_3^*, $\mathbf{H}_2^*$, $\mathbf{H}_3^*$, $\boldsymbol{\Gamma}^*$, Y^* are defined in the same way as their counterparts in $\boldsymbol{\psi}$.

EQUATIONS ON φ_i, $\mathbf{H}_i$, Y AND Γ

Upon substituting (2) and (26) in (7), we find

$$\nabla[\nabla^2(k_{11}\varphi_1 - k_{12}\varphi_2 - k_{13}\varphi_3) + \varphi_1^* - \rho\ddot{\varphi}_1] + \nabla\times[\nabla^2(\bar{k}_{11}\mathbf{H}_1 - \bar{k}_{12}\mathbf{H}_2 - \bar{k}_{13}\mathbf{H}_3) + \mathbf{H}_1^* - \rho\ddot{\mathbf{H}}_1] = 0, \quad (27a)$$

$$(\nabla\nabla)^D[k_{21}\varphi_1 + k_{22}\nabla^2\varphi_2 - k'_{22}\varphi_2 + k_{23}\nabla^2\varphi_3 + \tfrac{3}{2}\varphi_2^* - \tfrac{1}{2}\rho' d^2\ddot{\varphi}_2] + (\nabla\nabla\times)^S[\bar{k}_{21}\mathbf{H}_1 + \bar{k}_{22}\nabla^2\mathbf{H}_2 - \bar{k}'_{22}\mathbf{H}_2 + \bar{k}_{23}\nabla^2\mathbf{H}_3 + 2\mathbf{H}_2^* - \tfrac{2}{3}\rho' d^2\ddot{\mathbf{H}}_2] + (a_{10}+a_{13})\nabla^2\mathbf{\Gamma} - (b_2+b_3)\mathbf{\Gamma} + \mathbf{\Gamma}^* - \tfrac{1}{3}\rho' d^2\ddot{\mathbf{\Gamma}} = 0, \quad (27b)$$

$$\nabla^2(k_{31}\varphi_1 + k_{32}\nabla^2\varphi_2 + k_{33}\nabla^2\varphi_3 - k'_{33}\varphi_3 + 3\varphi_3^* - \rho' d^2\ddot{\varphi}_3) = 0, \quad (27c)$$

$$\nabla^2(\bar{k}_{31}\mathbf{H}_1 + \bar{k}_{32}\nabla^2\mathbf{H}_2 + \bar{k}_{33}\nabla^2\mathbf{H}_3 - \bar{k}'_{33}\mathbf{H}_3 + 2\mathbf{H}_3^* - \tfrac{2}{3}\rho' d^2\ddot{\mathbf{H}}_3) + \nabla[(a_{10}-a_{13})\nabla^2\mathrm{Y} - (b_2-b_3)\mathrm{Y} + \mathrm{Y}^* - \tfrac{1}{3}\rho' d^2\ddot{\mathrm{Y}}] = 0. \quad (27d)$$

Finally, as solutions of (27), we may take solutions of

$$\left.\begin{aligned} k_{11}\nabla^2\varphi_1 - k_{12}\nabla^2\varphi_2 - k_{13}\nabla^2\varphi_3 + \varphi_1^* &= \rho\ddot{\varphi}_1, \\ k_{21}\varphi_1 + k_{22}\nabla^2\varphi_2 - k'_{22}\varphi_2 + k_{23}\nabla^2\varphi_3 + \tfrac{3}{2}\varphi_2^* &= \tfrac{1}{2}\rho' d^2\ddot{\varphi}_2, \\ k_{31}\varphi_1 + k_{32}\nabla^2\varphi_2 + k_{33}\nabla^2\varphi_3 - k'_{33}\varphi_3 + 3\varphi_3^* &= \rho' d^2\ddot{\varphi}_3, \end{aligned}\right\} \quad (28)$$

$$\left.\begin{aligned} \bar{k}_{11}\nabla^2\mathbf{H}_1 - \bar{k}_{12}\nabla^2\mathbf{H}_2 - \bar{k}_{13}\nabla^2\mathbf{H}_3 + \mathbf{H}_1^* &= \rho\ddot{\mathbf{H}}_1, \\ \bar{k}_{21}\mathbf{H}_1 + \bar{k}_{22}\nabla^2\mathbf{H}_2 - \bar{k}'_{22}\mathbf{H}_2 + \bar{k}_{23}\nabla^2\mathbf{H}_3 + 2\mathbf{H}_2^* &= \tfrac{2}{3}\rho' d^2\ddot{\mathbf{H}}_2, \\ \bar{k}_{31}\mathbf{H}_1 + \bar{k}_{32}\nabla^2\mathbf{H}_2 + \bar{k}_{33}\nabla^2\mathbf{H}_3 - \bar{k}'_{33}\mathbf{H}_3 + 2\mathbf{H}_3^* &= \tfrac{2}{3}\rho' d^2\ddot{\mathbf{H}}_3, \end{aligned}\right\} \quad (29)$$

$$(a_{10}+a_{13})\nabla^2\mathbf{\Gamma} - (b_2+b_3)\mathbf{\Gamma} + \mathbf{\Gamma}^* = \tfrac{1}{3}\rho' d^2\ddot{\mathbf{\Gamma}}, \quad (30)$$

$$(a_{10}-a_{13})\nabla^2\mathrm{Y} - (b_2-b_3)\mathrm{Y} + \mathrm{Y}^* = \tfrac{1}{3}\rho' d^2\ddot{\mathrm{Y}}. \quad (31)$$

These are equations analogous to (3). The solution of (28)–(31) for plane waves is identical with the complete solution of (4), for plane waves, that was given in [1]. However, the general completeness of (28)–(31) has not been ascertained.

REFERENCE

[1] R. D. MINDLIN, *Arch. Ration. Mech. Analysis*, **16**, 51 (1964).

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Résumé—Dans la théorie de l'élasticité classique il est possible de réduire les équations du mouvement en déplacements à la forme de Lamé en utilisant les résolutions de Helmholtz. L'étude présente la réduction analogue pour la théorie d'un corps elastique isotrope avec micro-structure, évolue par l'auteur [1].

Абстракт—По классической теории эластичности возможно свести уравнения смещения для движения к форме Ламэ при помощи разрешения Гельмгольца.

В настоящей работе аналогичное преобразование сделано для уравнений, недавно выведенных [1], изотропного эластичного матерьяла обладающего микро-структурой.